

Basic Properties of Coherent and Generalized Coherent Operators Revisited

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Abstract

In this letter we make a brief review of some basic properties (the matrix elements, the trace, the Glauber formula) of coherent operators and study the corresponding ones for generalized coherent operators based on Lie algebra $\text{su}(1,1)$. We also propose some problems.

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1 Introduction

Coherent states or generalized coherent states play an important role in quantum physics, in particular, quantum optics, see [1] and its references. They also play an important one in mathematical physics. See the book [2]. For example, they are very useful in performing stationary phase approximations to path integral, [3], [4], [5].

Coherent operators which produce coherent states are very useful because they are unitary. They are also easy to handle. Why are they so handy ? The basic reason is probably that they are subject to the elementary Baker-Campbell-Hausdorff formula. Many basic properties of them have been investigated, see [1]. In this letter we are particularly interested in the following three ones : the matrix elements, the trace and the Glauber's formula. See [2] or [6] more recently .

Generalized coherent operators which produce generalized coherent states are also useful. But the corresponding properties have not been investigated as far as the author knows. The one of reasons is that they are not so easy to handle. Of course we have the disentangling formula corresponding to the elementary Baker-Campbell-Hausdorff formula, but they are not handy too.

In this letter we investigate the three properties above for generalized coherent operators based on Lie algebra $su(1,1)$. But we don't treat generalized coherent operators based on Lie algebra $su(2)$ in this one. We leave this case to the readers.

By the way we are very interested in Geometric Quantum Computer, in particular, Holonomic Quantum Computer which has been proposed by Zanardi and Rasetti [7], [8] and developed by Fujii [9], [10], [11] and Pachos and Chountasis [12]. As a general introduction to Quantum Computer (Computation) [13] or [14] are recommended.

The hidden aim of this letter is to apply some results in this letter to Theory of Quantum Computer, in particular , quantum measurement or quantum entanglement or etc. See [15] and its references.

The details of this letter will be published elsewhere [16].

2 Coherent and Generalized Coherent Operators

Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N \equiv a^\dagger a$ (: number operator), then

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a^\dagger, a] = -\mathbf{1}. \quad (1)$$

Let \mathcal{H} be a Fock space generated by a and a^\dagger , and $\{|n\rangle \mid n \in \mathbf{N} \cup \{0\}\}$ be its basis. The actions of a and a^\dagger on \mathcal{H} are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad N|n\rangle = n|n\rangle \quad (2)$$

where $|0\rangle$ is a normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0\rangle = 1$). From (2) state $|n\rangle$ for $n \geq 1$ are given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \quad (3)$$

These states satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}. \quad (4)$$

Let us state coherent states. For the normalized state $|z\rangle \in \mathcal{H}$ for $z \in \mathbf{C}$ the following three conditions are equivalent :

$$(i) \quad a|z\rangle = z|z\rangle \quad \text{and} \quad \langle z|z\rangle = 1 \quad (5)$$

$$(ii) \quad |z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle = e^{-|z|^2/2} e^{za^\dagger} |0\rangle \quad (6)$$

$$(iii) \quad |z\rangle = e^{za^\dagger - \bar{z}a} |0\rangle. \quad (7)$$

In the process from (6) to (7) we use the famous elementary Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B \quad (8)$$

whenever $[A, [A, B]] = [B, [A, B]] = 0$, see [1]. This is the key formula.

Definition The state $|z\rangle$ that satisfies one of (i) or (ii) or (iii) above is called the coherent state.

The important feature of coherent states is the following partition (resolution) of unity.

$$\int_{\mathbf{C}} \frac{[d^2 z]}{\pi} |z\rangle\langle z| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}, \quad (9)$$

where we have put $[d^2 z] = d(\operatorname{Re} z)d(\operatorname{Im} z)$ for simplicity.

Since the operator

$$U(z) = e^{za^\dagger - \bar{z}a} \quad \text{for } z \in \mathbf{C} \quad (10)$$

is unitary, we call this a (unitary) coherent operator. For these operators the following properties are crucial. For $z, w \in \mathbf{C}$

$$U(z)U(w) = e^{z\bar{w} - \bar{z}w} U(w)U(z), \quad (11)$$

$$U(z+w) = e^{-\frac{1}{2}(z\bar{w} - \bar{z}w)} U(z)U(w). \quad (12)$$

In the following section we list several basic properties of this operator.

Next let us state generalized coherent states. Let $\{k_+, k_-, k_3\}$ be a Weyl basis of Lie algebra $su(1, 1) \subset sl(2, \mathbf{C})$,

$$k_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad k_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad k_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (13)$$

Then we have

$$[k_3, k_+] = k_+, \quad [k_3, k_-] = -k_-, \quad [k_+, k_-] = -2k_3. \quad (14)$$

We note that $(k_+)^\dagger = -k_-$.

Next we consider a spin K (> 0) representation of $su(1, 1) \subset sl(2, \mathbf{C})$ and set its generators $\{K_+, K_-, K_3\}$ ($(K_+)^\dagger = K_-$ in this case),

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \quad (15)$$

We note that this (unitary) representation is necessarily infinite dimensional. The Fock space on which $\{K_+, K_-, K_3\}$ act is $\mathcal{H}_K \equiv \{|K, n\rangle | n \in \mathbf{N} \cup \{0\}\}$ and whose actions are

$$\begin{aligned} K_+ |K, n\rangle &= \sqrt{(n+1)(2K+n)} |K, n+1\rangle, \\ K_- |K, n\rangle &= \sqrt{n(2K+n-1)} |K, n-1\rangle, \\ K_3 |K, n\rangle &= (K+n) |K, n\rangle, \end{aligned} \quad (16)$$

where $|K, 0\rangle$ is a normalized vacuum ($K_-|K, 0\rangle = 0$ and $\langle K, 0|K, 0\rangle = 1$). We have written $|K, 0\rangle$ instead of $|0\rangle$ to emphasize the spin K representation, see [3]. From (14), states $|K, n\rangle$ are given by

$$|K, n\rangle = \frac{(K_+)^n}{\sqrt{n!(2K)_n}} |K, 0\rangle, \quad (17)$$

where $(a)_n$ is the Pochhammer's notation

$$(a)_n \equiv a(a+1) \cdots (a+n-1). \quad (18)$$

These states satisfy the orthogonality and completeness conditions

$$\langle K, m|K, n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |K, n\rangle \langle K, n| = \mathbf{1}_K. \quad (19)$$

Now let us consider a generalized version of coherent states :

Definition We call a state

$$|z\rangle \equiv e^{zK_+ - \bar{z}K_-} |K, 0\rangle \quad \text{for } z \in \mathbf{C}. \quad (20)$$

the generalized coherent state (or the coherent state of Perelomov's type based on $su(1, 1)$ in our terminology).

This is the extension of (7). See the book [2].

Then the partition of unity corresponding to (9) is

$$\begin{aligned} & \int_{\mathbf{C}} \frac{2K-1}{\pi} \frac{\tanh(|z|)[d^2z]}{(1 - \tanh^2(|z|))|z|} |z\rangle \langle z| \\ &= \int_{\mathbf{D}} \frac{2K-1}{\pi} \frac{[d^2\zeta]}{(1 - |\zeta|^2)^2} |\zeta\rangle \langle \zeta| = \sum_{n=0}^{\infty} |K, n\rangle \langle K, n| = \mathbf{1}_K, \end{aligned} \quad (21)$$

where

$$\mathbf{C} \rightarrow \mathbf{D} : z \mapsto \zeta = \zeta(z) \equiv \frac{\tanh(|z|)}{|z|} z \quad \text{and} \quad |\zeta\rangle \equiv (1 - |\zeta|^2)^K e^{\zeta K_+} |K, 0\rangle. \quad (22)$$

In the process of the proof we use the disentangling formula :

$$e^{zK_+ - \bar{z}K_-} = e^{\zeta K_+} e^{\log(1-|\zeta|^2)K_3} e^{-\bar{\zeta}K_-} = e^{-\bar{\zeta}K_-} e^{-\log(1-|\zeta|^2)K_3} e^{\zeta K_+}. \quad (23)$$

This is also the key formula for generalized coherent operators. See [2] or [17].

Here let us construct the spin K -representations making use of Schwinger's boson method.

First we assign

$$K_+ \equiv \frac{1}{2} (a^\dagger)^2, \quad K_- \equiv \frac{1}{2} a^2, \quad K_3 \equiv \frac{1}{2} \left(a^\dagger a + \frac{1}{2} \right). \quad (24)$$

Then we have

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \quad (25)$$

That is, the set $\{K_+, K_-, K_3\}$ gives a unitary representation of $su(1, 1)$ with spin $K = 1/4$ and $3/4$, [2]. Now we also call an operator

$$S(z) = e^{\frac{1}{2}\{z(a^\dagger)^2 - \bar{z}a^2\}} \quad \text{for } z \in \mathbf{C} \quad (26)$$

the squeezed operator, see the papers in [1] or the book [2].

Next we consider the system of two-harmonic oscillators. If we set

$$a_1 = a \otimes 1, \quad a_1^\dagger = a^\dagger \otimes 1; \quad a_2 = 1 \otimes a, \quad a_2^\dagger = 1 \otimes a^\dagger, \quad (27)$$

then it is easy to see

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2. \quad (28)$$

We also denote by $N_i = a_i^\dagger a_i$ number operators.

Now we can construct representation of Lie algebras $su(2)$ and $su(1, 1)$ making use of Schwinger's boson method, see [3], [4]. Namely if we set

$$su(2): \quad J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_3 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), \quad (29)$$

$$su(1, 1): \quad K_+ = a_1^\dagger a_2^\dagger, \quad K_- = a_2 a_1, \quad K_3 = \frac{1}{2} (a_1^\dagger a_1 + a_2^\dagger a_2 + 1), \quad (30)$$

then we have

$$su(2): \quad [J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3, \quad (31)$$

$$su(1, 1): \quad [K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \quad (32)$$

In the following we define (unitary) generalized coherent operators based on Lie algebras $su(2)$ and $su(1, 1)$.

Definition We set

$$su(2) : \quad W(z) = e^{za_1^\dagger a_2 - \bar{z}a_2^\dagger a_1} \quad \text{for } z \in \mathbf{C}, \quad (33)$$

$$su(1, 1) : \quad V(z) = e^{za_1^\dagger a_2^\dagger - \bar{z}a_2 a_1} \quad \text{for } z \in \mathbf{C}. \quad (34)$$

For the details of $W(z)$ and $V(z)$ see [2] and [3], or [10] and [12].

In the section 4 we study the basic properties (corresponding to those of coherent operators in section 3) of the generalized coherent operators.

Before ending this section let us ask a question.

What is a relation between (34) and (26) of generalized coherent operators based on $su(1, 1)$?

The answer is given by Paris [15]:

The Formula We have

$$W(-\frac{\pi}{4})S_1(z)S_2(-z)W(-\frac{\pi}{4})^{-1} = V(z), \quad (35)$$

where $S_j(z) = (26)$ with a_j instead of a .

Namely, $V(z)$ is given by “rotating” the product $S_1(z)S_2(-z)$ by $W(-\frac{\pi}{4})$. This is an interesting relation.

3 Basic Properties of Coherent Operators

We make a brief review of some basic properties of coherent operators (10). For the elegant proofs see the book [2], or the paper [6] and its references.

The Matrix Elements The matrix elements of $U(z)$ are :

$$(i) \quad n \leq m \quad \langle n|U(z)|m \rangle = e^{-\frac{1}{2}|z|^2} \sqrt{\frac{n!}{m!}} (-\bar{z})^{m-n} L_n^{(m-n)}(|z|^2), \quad (36)$$

$$(ii) \quad n \geq m \quad \langle n|U(z)|m \rangle = e^{-\frac{1}{2}|z|^2} \sqrt{\frac{m!}{n!}} z^{n-m} L_m^{(n-m)}(|z|^2), \quad (37)$$

where $L_n^{(\alpha)}$ is the associated Laguerre's polynomial defined by

$$L_n^{(\alpha)}(x) = \sum_{j=0}^n (-1)^j \binom{n+\alpha}{n-j} \frac{x^j}{j!}. \quad (38)$$

In particular $L_n^{(0)}$ is the usual Laguerre's polynomial and these are related to diagonal elements of $U(z)$. Here let us list the generating function and orthogonality condition of associated Laguerre's polynomials :

$$\frac{e^{-xt/(1-t)}}{(1-t)^{\alpha+1}} = \sum_{j=0}^{\infty} L_n^{(\alpha)}(x) t^j \quad \text{for } |t| < 1, \quad (39)$$

$$\int_0^{\infty} e^{-x} x^{\alpha} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{nm} \quad \text{for } \text{Re}(\alpha) > -1. \quad (40)$$

The Trace The Trace of $U(z)$ is

$$\text{Tr} U(z) = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} L_n^{(0)}(|z|^2)$$

from (36). Then taking a limit $t \rightarrow 1$ in (39) from the below we can see easily

$$\sum_{n=0}^{\infty} L_n^{(0)}(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

From this we can guess $\text{Trace } U(z) = \text{some } \delta \text{ function}$. In fact we have

$$\text{Tr} U(z) = \pi \delta^2(z) \equiv \pi \delta(x) \delta(y) \quad \text{if } z = x + iy. \quad (41)$$

For the proof we use (9)

$$\text{Tr} U(z) = \text{Tr}(U(z) \mathbf{1}) = \int_{\mathbf{C}} \frac{[d^2 w]}{\pi} \langle w | U(z) | w \rangle = \int_{\mathbf{C}} \frac{[d^2 w]}{\pi} \langle 0 | U(w)^{-1} U(z) U(w) | 0 \rangle. \quad (42)$$

It is easy to calculate the right hand side of (42) making use of (12) to get (41).

The Glauber's Formula The typical feature of coherent operators is the following Glauber's formula : Let A be any observable. Then we have

$$A = \int_{\mathbf{C}} \frac{[d^2 z]}{\pi} \text{Tr}[A U^{\dagger}(z)] U(z) \quad (43)$$

This formula plays an important role in the field of homodyne tomography, see [6].

4 Basic Properties of Generalized Coherent Operators

We in this section study the properties of generalized coherent operators (34) corresponding to ones of coherent operators. The some results may be known, but the author could not find such references in spite of his efforts. At any rate let us list our results.

In this section we take

The Assumption we assume that

$$2K \in \mathbf{N} \quad (44)$$

because we use a differential one of spin K representation of the group $SU(1, 1)$ as the representation in (15).

The Matrix Elements The matrix elements of $V(z)$ are :

$$(i) \quad n \leq m \quad \langle K, n | V(z) | K, m \rangle = \sqrt{\frac{n!m!}{(2K)_n(2K)_m}} \frac{(-\bar{\kappa})^{m-n}}{(1+|\kappa|^2)^{K+\frac{n+m}{2}}} \times \\ \sum_{j=0}^n (-1)^{n-j} \frac{(2K-1+m-j+n-j+j)!}{(2K-1)!(m-j)!(n-j)!j!} (1+|\kappa|^2)^j (|\kappa|^2)^{n-j}, \quad (45)$$

$$(ii) \quad n \geq m \quad \langle K, n | V(z) | K, m \rangle = \sqrt{\frac{n!m!}{(2K)_n(2K)_m}} \frac{\kappa^{n-m}}{(1+|\kappa|^2)^{K+\frac{n+m}{2}}} \times \\ \sum_{j=0}^m (-1)^{m-j} \frac{(2K-1+m-j+n-j+j)!}{(2K-1)!(m-j)!(n-j)!j!} (1+|\kappa|^2)^j (|\kappa|^2)^{m-j}, \quad (46)$$

where

$$\kappa \equiv \frac{\sinh(|z|)}{|z|} z = \cosh(|z|) \zeta. \quad (47)$$

A comment is in order. The author doesn't know whether or not the right hand sides of (45) and (46) could be written making use of some special functions such as generalized Laguerre's functions in (36) and (37).

The Trace The Trace of $V(z)$ is

$$\text{Tr}V(z) = \sum_{n=0}^{\infty} \left\{ \frac{\frac{n!}{(2K)_n}}{(1 + |\kappa|^2)^{K+n}} \sum_{j=0}^n (-1)^{n-j} \frac{(2K-1+2(n-j)+j)!}{(2K-1)!(n-j)!^2 j!} (1 + |\kappa|^2)^j (|\kappa|^2)^{n-j} \right\}$$

from (45) or (46), but it seems not easy to get a compact form (at least to the author).

As an another method we use (42). From (20) and (21) we have

$$\begin{aligned} \text{Tr}V(z) &= \text{Tr}(V(z)\mathbf{1}_{\mathbf{K}}) = \int_{\mathbf{C}} \frac{2K-1}{\pi} \frac{\tanh(|w|)[d^2w]}{(1 - \tanh^2(|w|)) |w|} \langle w|V(z)|w \rangle \\ &= \int_{\mathbf{C}} \frac{2K-1}{\pi} \frac{\tanh(|w|)[d^2w]}{(1 - \tanh^2(|w|)) |w|} \langle K, 0|V(w)^{-1}V(z)V(w)|K, 0 \rangle. \end{aligned} \quad (48)$$

We can perform the calculation of (48) in spite of very hard one.

Making use of change of variables $w \mapsto \zeta = \zeta(w) \equiv \frac{\tanh(|w|)}{|w|}w$ in (22) and disentangling formula (23) we have

$$\text{Tr}V(z) = \frac{1}{\{\cosh(|z|)\}^{2K}} \int_{\mathbf{D}} \frac{2K-1}{\pi} \frac{[d^2\zeta]}{(1 - |\zeta|^2)^2} \frac{1}{\left\{1 - \frac{\tanh(|z|)}{|z|} \frac{\bar{z}\zeta - z\bar{\zeta}}{1 - |\zeta|^2}\right\}^{2K}}. \quad (49)$$

Moreover we set $\chi \equiv \frac{\tanh(|z|)}{|z|}z$ for simplicity, then we have

$$\text{Tr}V(z) = (1 - |\chi|^2)^K \int_{\mathbf{D}} \frac{2K-1}{\pi} \frac{[d^2\zeta]}{(1 - |\zeta|^2)^2} \frac{1}{\left(1 - \frac{\bar{\chi}\zeta - \chi\bar{\zeta}}{1 - |\zeta|^2}\right)^{2K}}. \quad (50)$$

To calculate this we moreover assume $2K \in \mathbf{N} - \{1\}$ because the measure of the integral contains the term $2K - 1 = 0$!.

Under this assumption we can perform the integral and obtain

$$\int_{\mathbf{D}} \frac{2K-1}{\pi} \frac{[d^2\zeta]}{(1 - |\zeta|^2)^2} \frac{1}{\left(1 - \frac{\bar{\chi}\zeta - \chi\bar{\zeta}}{1 - |\zeta|^2}\right)^{2K}} = \frac{1}{2|\chi|(1 + |\chi|)^{2K-1}}. \quad (51)$$

As a result we have

$$\text{Tr}V(z) = \frac{(1 - |\chi|^2)^K}{2|\chi|(1 + |\chi|)^{2K-1}} = \frac{1 + |\chi|}{2|\chi|} \left(\frac{1 - |\chi|}{1 + |\chi|} \right)^K. \quad (52)$$

A comment is in order. The author believes that

Conjecture we have for all $2K \geq 1$

$$\int_{\mathbf{D}} \frac{2K-1}{\pi} \frac{[d^2\zeta]}{(1 - |\zeta|^2)^2} \frac{1}{\left(1 - \frac{\bar{\chi}\zeta - \chi\bar{\zeta}}{1 - |\zeta|^2}\right)^{2K}} = \frac{1}{2|\chi|(1 + |\chi|)^{2K-1}}. \quad (53)$$

Namely we believe that the formula (52) holds for all $2K \geq 1$.

The Glauber's Formula Unfortunately in this case we have no Glauber's formula :

For a observable A

$$A \neq \int_{\mathbf{C}} \frac{2K-1}{\pi} \frac{\tanh(|z|)[d^2z]}{(1 - \tanh^2(|z|))|z|} \text{Tr}[AV^\dagger(z)]V(z) \quad (54)$$

In fact if we set $A = |K, 0\rangle\langle K, 0|$ then it is not difficult to check (54) making use of (45) and (46).

5 Discussion

In this paper we listed the three basic properties of coherent operators and investigated these ones for generalized coherent operators based on $su(1, 1) \cdots V(z)$ in (33). But we have not investigated these for generalized coherent operators based on $su(2) \cdots W(z)$ in (34). We leave these to the readers.

The several caluculations performed in Section 4 are not easy. We will publish these ones in [16].

Acknowledgment.

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[Note added in proof]

From (52) with $|\chi| = \tanh(|z|)$ it is easy to see

$$\text{The right hand side of (52)} = \frac{e^{-2|z|K}}{1 - e^{-2|z|}} = \sum_{n=0}^{\infty} e^{-2|z|(K+n)} \quad (55)$$

, so that from (16) we can conjecture $\text{Tr}V(z) = \text{Tr} e^{-2|z|K_3}$ (this was pointed out by S. Sakoda. The author thanks him for this suggestion.) What is this meaning ? After some considerations we reached

Formula we have the (new ?) decomposition formula

$$e^{zK_+ - \bar{z}K_-} = e^{\frac{\pi}{4}\left(\frac{z}{|z|}K_+ + \frac{\bar{z}}{|z|}K_-\right)} e^{-2|z|K_3} e^{-\frac{\pi}{4}\left(\frac{z}{|z|}K_+ + \frac{\bar{z}}{|z|}K_-\right)} \quad (56)$$

for all $2K \geq 1$ (!).

A comment is in order. As far as the author knows this decomposition formula has not been used in the references.

Therefore we obtain

$$\text{Tr} V(z) = \text{Tr } e^{-2|z|K_3} = \text{The right hand side of (52)} \quad (57)$$

for all $2K \geq 1$. Namely we could solve our conjecture (53) indirectly. But the direct proof is still unknown.

Problem Prove directly for all $2K \geq 1$

$$\int_{\mathbb{D}} \frac{2K-1}{\pi} \frac{[d^2\zeta]}{(1-|\zeta|^2)^2} \frac{1}{\left(1 - \frac{\bar{\chi}\zeta - \chi\bar{\zeta}}{1-|\zeta|^2}\right)^{2K}} = \frac{1}{2|\chi|(1+|\chi|)^{2K-1}}.$$

We note that in the case $2K \in \mathbf{N} - \{1\}$ we have used the residue theorem in Complex Analysis.

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